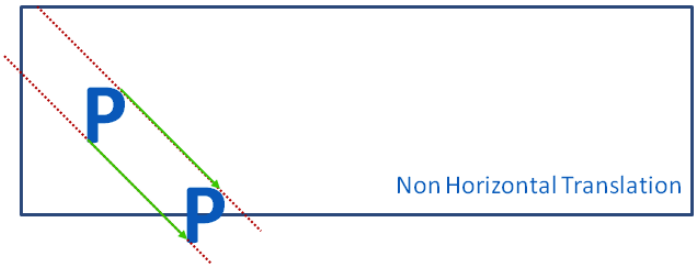
The mathematics of friezes explained

A frieze is defined as the repetition of a design along a unique direction. Mathematically, the isometries of some figure F form a group (for group definition see, for instance, http://en.wikipedia.org/wiki/Group_theory). In this way, any isometry *g* is an element of the set of isometries of the plane that maps the frieze on itself. It can be shown that there are only four isometric operations on the plane: translation, reflection, rotation and glide reflection. See http://www.math.uga.edu/~clint/2005/5210/isom.htm.

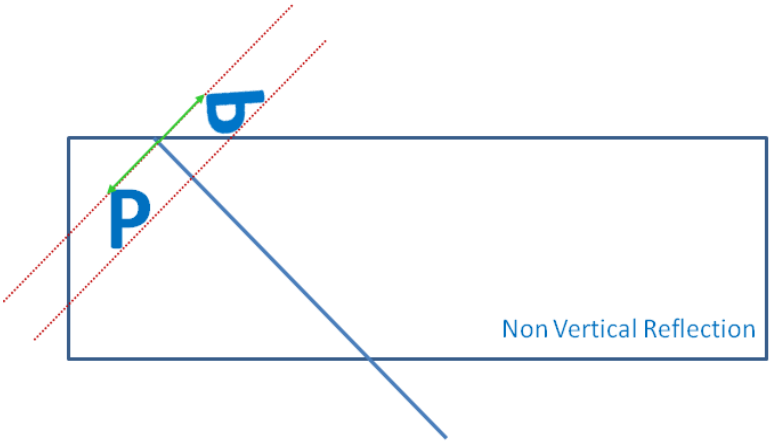
1.1 Translation

When applied to a band (in order to produce a frieze), the translation direction must be parallel to the sides of the band not as shown below.



1.2 Vertical reflection

When applied to a band (in order to produce a frieze), the reflection axis must be perpendicular to the sides of the band not as shown below. Notice that for these non vertical reflections the property that the frieze will be mapped on itself does not hold.

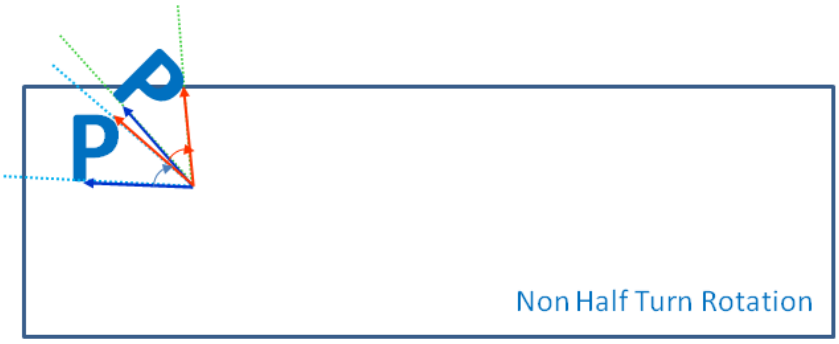


1.3 Horizontal reflection

When applied to a band (in order to produce a frieze), the reflection axis must be in the middle of the band and parallel to its sides not as shown below. This is to enable the frieze to be mapped onto itself.

1.4 Rotation

Due to the fact that frieze isometries must map the frieze on itself, only half turn rotations are allowed, not as shown below.



2. Composition of isometries

Mathematically, it is shown by group theory arguments that the possible compositions of isometries produce only 7 different frieze patterns. The classification of the isometries exhibited by the 7 different friezes is given by the following table:

Friezes Pattern	Translation	Vertical Reflection	Horizontal Reflection	Glide Reflection	Rotation
pppp	•				
pqpq	•	•			
pppp bbbb	•		•	•	
p p p b b b	•			•	
p p p d d d	•				•
pq pq bd	•	•		•	•
pqpqpq bdbdbd	•	•	•	•	•

3. Additional materials

Additional materials may be found at:
http://www.apm.pt/files/_SP_Vieira_Ferreira_Mamede_4a4dce3ea12b6.pdf (document in portuguese)